MATHEMATICS AND THE SEARCH FOR KNOWLEDGE

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I

The Failings of the Senses and Intuition

Sense perceptions are sense deceptions. Descartes

Despite the denials of Berkeley, the qualifications of Hume, and the reservations of Heraclitus, Plato, Kant, and Mill concerning what we can know about the external world, physicists and mathematicians do believe that there is an external world. They would argue that even if all human beings were suddenly wiped out, the external or physical world would continue to exist. When a tree crashes to the ground in a forest, a sound is created even if no one is there to hear it. We have five senses—sight, hearing, touch, taste, and smell—and each of these constantly receives messages from this external world. Whether or not our sensations are reliable, we do receive them from some external source.

For practical reasons, such as remaining alive and possibly for improving life in the external world, we certainly want to know as much as possible about this world. We must distinguish land from sea. We must grow food, build shelters, and protect ourselves against wild beasts. Why then can we not rely on our senses to achieve these aims? Primitive civilizations have done so. But just as to the pure in heart the world is pure, so to the simpleminded the world is simple.

In attempting to improve our material way of life, we were forced to extend our knowledge of the external world. Thus we necessarily extended the uses of our senses to the utmost. Unfortunately for us, our senses are not only limited but also deceiving. Trusting solely in the senses can even lead to disaster. Let us note some of these limitations.

Of the five senses, the sense of sight is perhaps the most valuable.

Let us first test how much we can depend on this sense. We begin with a few simple examples. Through the years, many deceptive visual figures were deliberately conceived and constructed to show the limitations of the eye. In fact, physicists and astronomers of the nineteenth century took a lively interest in visual illusions because they were concerned that visual observations might prove to be unreliable. Wilhelm Wundt, an assistant to the famous physiologist, physician, and scientist Hermann von Helmholtz (1821–1894), constructed Figure 1. The illusion is simply that the vertical line looks longer than the horizontal line, which is of equal length. This illusion can be reversed. In Figure 2 the height and width seem equal, but the width is larger.

FIGURE 1

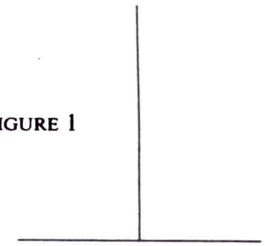
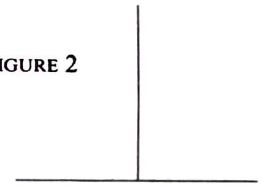


FIGURE 2



The illusion in Figure 3 was devised by Franz Müller-Lyer in 1899 and is also known as the Ernst Mach illusion. The two horizontal lines are actually the same length.

FIGURE 3

In Figure 4 the dot is at the middle of the horizontal line. Both illusions are created by angles.

FIGURE 4

In Figure 5 the length of the top horizontal line of the lower figure appears shorter than that of the top line of the upper figure. Incidentally, it is difficult to believe that the maximum horizontal width of the lower figure is as great as the maximum height of that figure.

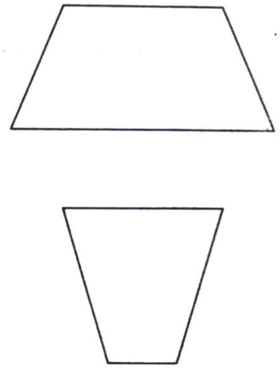


FIGURE 5

A striking illusion involving the influence of angles is shown in Figure 6. Here the diagonals *AB* and *AC* of the two parallelograms are of equal length, but the one on the right appears much shorter.

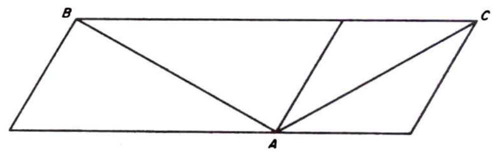
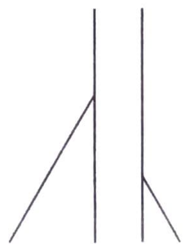


FIGURE 6

If oblique lines are extended across vertical ones, as in Figure 7, the illusion is very striking. The oblique line on the right if extended would meet the upper end of the oblique line on the left; however, the *apparent* point of intersection is somewhat lower. This well-known illusion is attributed to Johann P. Poggendorf (about 1860).

FIGURE 7



In Figure 8 the three horizontal lines are of equal length, although they appear unequal. This illusion is primarily a result of the sizes of the angles made by the lines at the ends. Within certain limits, the greater the angle the greater is the apparent elongation of the central horizontal portion.

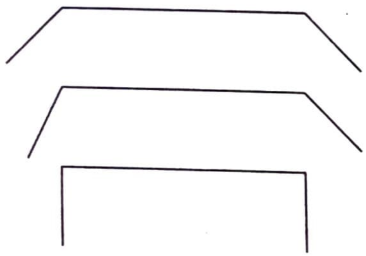


FIGURE 8

A striking illusion of contrast is shown in Figure 9. The central circles of the two figures are equal, although the one surrounded by the large circles appears much smaller than the one surrounded by the smaller circles.

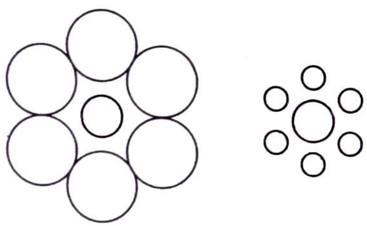


FIGURE 9

Another mechanism is believed to be at work in the Müller-Lyer illusion. In the left drawing of Figure 10 the horizontal lines at each end of the vertical line *A* are interpreted as the upper and the lower edges of two walls meeting at a corner. In this case the vertical line would be interpreted as being the foreground of a real-world scene. In the right drawing of Figure 10 the horizontal lines are again seen as

wall edges, but in this case they appear to be converging on an inside corner. As a result the vertical line *B* is interpreted as being in the background. The judgment of size constancy then enlarges the perceived length of line *B* and diminishes line *A*.

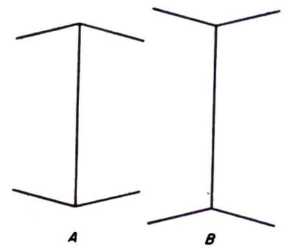


FIGURE 10

Johann Z. Zöllner was the first to describe the illusion illustrated in Figures 11 and 12. Zöllner accidentally noticed the illusion on a pattern designed for dress goods. The long parallel lines in Figure 11 appear to diverge, whereas in Figure 12 they appear to converge.

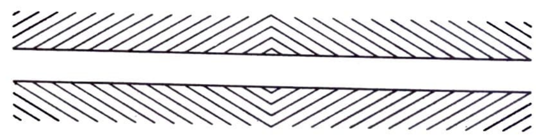


FIGURE 11

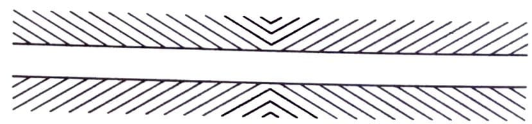


FIGURE 12

In the Hering illusion (Figure 13), published in 1861 by Ewald Hering, the straight horizontal lines acquire the illusion of curving in relation to the converging oblique lines.

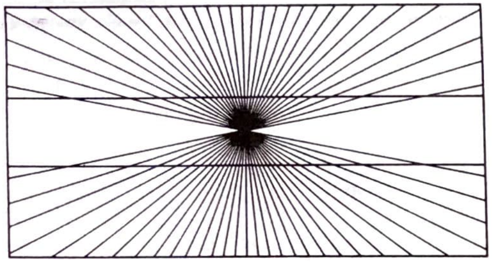
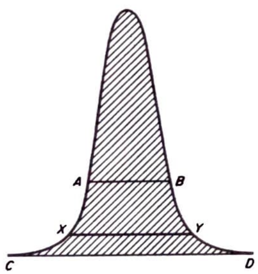


FIGURE 13

The unreliability of vision is demonstrated by still another example devised by Professor S. Tolansky. In Figure 14, commonly used in statistical studies, the baseline *CD* is as long as the height of the figure. Moreover, a viewer, when asked to draw a line across the curve that is half the width of the baseline, would almost surely choose the line *AB*. However, the correct choice is *XY*.

FIGURE 14

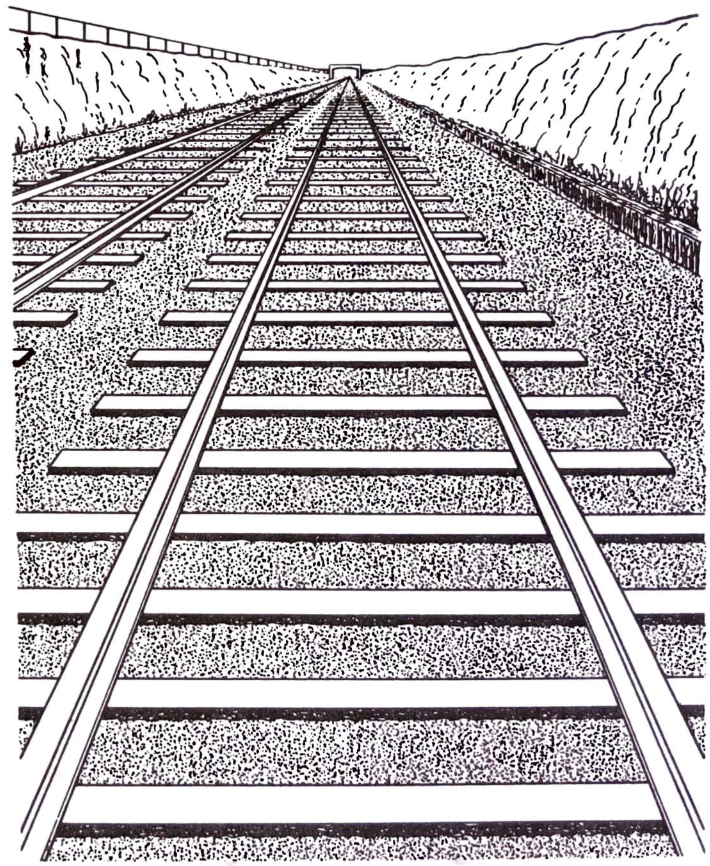


We are all familiar with an illusion that is deliberately and skillfully fashioned, namely, realistic painting. The intent is to present a three-dimensional scene on a flat or two-dimensional canvas. One of the great achievements of the Renaissance painters was to devise a mathematical scheme, known as linear perspective, which achieves the desired illusion.

Some simple examples of the illusion of linear perspective are part of our everyday experience. The principle involved in these examples

and in the theory of linear perspective is that lines in the actual scene that recede directly from the observer must appear to come together at a distant point called the vanishing point. A simple example is furnished by the appearance of two parallel railroad tracks (Figure 15) that appear to meet at a distant point.

FIGURE 15



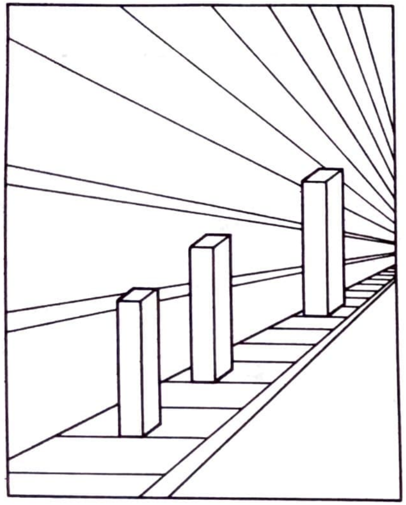


FIGURE 16

FIGURE 17

The influence of perspective is particularly apparent in Figure 16, where the usual perspective lines are drawn to suggest a scene. The tall boxes are of the same size and physical dimensions, but the farthest one actually appears much larger. Experience, expecting a diminution of size with increasing distance, actually causes the box on the right to appear larger than it really is.

We allow ourselves to be deceived and even enjoy the deception when we admire realistic painting. Such paintings must of necessity be two-dimensional, but if they are executed in accordance with the laws of linear mathematical perspective, we believe we are looking at a three-dimensional scene. Raphael's *School of Athens* (Figure 17) is a good example.

Of course, the mathematical system of linear perspective takes advantage of optical illusions. Objects or people that are intended to appear more distant than those in the foreground are drawn smaller, which is how the human eye would see them. Artists also take advantage of another optical effect, the loss of intensity or brightness of a distant object.

There are other visual illusions in our daily experiences. The sun and the moon seem larger when on the horizon than when overhead, because we unconsciously allow for our belief that they are closer when on the horizon. Precise measurement would of course show that their size remains constant.

If we measure the angle subtended at the eye by the diameter of the moon, we shall find it to be almost exactly one-half a degree. Because the whole semicircular vault of the heavens subtends 180 degrees, the angle subtended by the moon is a mere 1/360 of the vault of the heavens. The proportionate *area* occupied by the moon is the astonishingly small amount of only 1/100,000 of the heavenly vault, but if we consider how striking an object the full moon is, it is hard to appreciate how small a fraction of the sky area it occupies.

A number of other illusions involve what is known as the refraction or bending of light. All of us have noticed that a stick partially immersed in water seems bent and that the bending occurs at the surface of the water.

An aerial refraction phenomenon that has attracted attention since ancient times is the *mirage*, a phenomenon that comes about by the effect of variation in air density produced by the heat of the sun, combined with a total reflection effect. A simple mirage with which most of us are familiar occurs when one travels along a long, straight, flat road on a hot summer day. Well ahead the road appears to be covered with water, yet on traveling further one finds that the road is quite dry. Let us consider what causes this effect.

The effect only appears when the road surface is heated strongly by the sun. The air in contact with the road thus warms up, which lowers its density and causes it to rise continuously. It follows that the refraction of light is also less at the bottom than in the top layers of air. Let us imagine, as in Figure 18, that there is a succession of changing layers. Light passes through the layers and comes to our eyes from

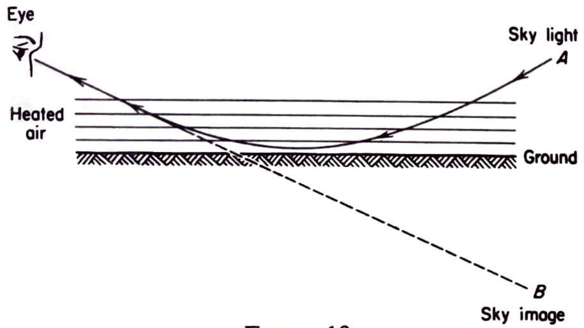


FIGURE 18

low down near the ground. As a result, the observer sees light from the sky originating at *A*, but it seems to come from *B*. This is precisely what would have happened if there had been a pool of water on the ground, for with wet ground we see the reflection of the sky's light. The effect of the heating of the road has therefore led to an apparent reflection of the kind that we always associate with water on the road. We are fooled and think the road is actually wet.

Most of the illusions we have examined were deliberately devised by psychologists, but we need not resort to contrived figures to appreciate that our sense of sight is constantly in error—and for understandable reasons. Because of the refraction or bending of light in the Earth's atmosphere, the sun is visible even when it is below the horizon. The Earth seems flat, and the sun seems to move around an apparently stationary Earth that does not even seem to rotate. Suppose the sun is high in the sky. To the question, "Do you see the sun now?" we would immediately answer yes. Yet the light from the sun takes eight minutes to reach us and in those eight minutes the sun could have exploded. When the sun is low on the horizon it does not appear as a circular disk but instead is somewhat flattened; its vertical diameter appears to be shortened. This phenomenon is caused by the bending of the light

rays as they traverse the Earth's atmosphere. The stars, because they are so far away, seem like specks of light.

Visual distortions are often called illusions, but "illusions" come in many different forms. Color is transmitted to the brain from the retina through three channels. Three types of color receptors (cones) exist, each one sensitive to one of the three primary colors: red, green, or blue. White light activates all three color channels. Every object absorbs some light rays and reflects others. The color we see is what is reflected. A white object reflects all the light that falls on it. Is then a brown table actually brown? The flame from a candle in a brightly lit room seems dim, but in a dark room it seems bright. A piece of wood seems solid, but it is really a collection of atoms held together by interatomic forces. The hardness is not that of a continuous substance.

There may be distortions in other types of sensations: temperature, taste, the loudness or pitch of sound, and the speed at which objects appear to move. Let us consider an illusion of temperature. Immerse one hand in a bowl of hot water, the other in a bowl of cold water. After a few minutes, immerse both hands in a bowl of tepid water. Although both hands are now in water of the same temperature, the one that was in hot water feels the tepid as cold while the other feels it as hot. It is interesting to observe that if a hand is placed in water that is then heated (or cooled) gradually so that the change of temperature is not felt, the hand still adapts to the changed temperature.

The taste sense is also subject to several illusions. Sweet drinks taste gradually less sweet. Try keeping a strong solution of water and sugar in the mouth for a few seconds and then taste fresh water. It will now taste distinctly salty.

Misapprehensions of speed are common. A car moving at thirty miles per hour seems almost ridiculously slow after half an hour's continuous driving at highway speed. A very common illusion is presented by two trains in a station. If your train is stationary and the other is moving, you can readily be deceived into thinking that your train is also moving.

Some distortions are caused by the sensory receptors becoming fatigued or adapted by prolonged or intense stimulation. This can happen with any one of the senses and can result in considerable distortions. An example is the illusions of weight. After carrying a heavy weight for a few minutes, any much lighter weight will seem to be far lighter in proportion to what it actually is.

Beyond the illusions wherein we actually sense physical objects or happenings, we must take into account that our senses are limited. The normal human ear can hear sounds whose frequencies are about 20 to 20,000 cycles per second. The normal human eye can receive light whose wavelengths (see Chapter VII) range from 16 to 32 millionths of an inch. Yet both sound and light (strictly in the latter case, electromagnetic waves) exist and are physically real much beyond the ranges we can perceive with our senses alone. Even white light is not white but, as Newton showed, a composite of many frequencies. The eye registers only the composite. In fact, there are no colors in the physical world. As Goethe put it, color is what we see.

We never directly perceive a physical object but a sense datum. Our senses present and will always present not the faithful image of an objective reality, whether or not beyond our reach, but rather the image of the relationship between man and reality.

Humans claim, however, that beyond the senses we have intuition that can surely be relied on. Let us see how reliable human intuition is.

Suppose a man drives from New York City to Buffalo (a distance of 400 miles) at 60 miles per hour, and then drives back at 30 miles per hour. What is his average speed? Intuition almost certainly tells us that the average speed is 45 miles per hour. The correct answer, obtained by dividing the total distance by the total time, is about 40 miles per hour.

Let us consider some more examples of the reputed power of intuition. Suppose we put P dollars in the bank at a compound interest of i percent and keep it there until the total amount doubles. Let us suppose this happens in n years. It would seem reasonable to assume, if one put in $2P$ dollars at the same interest rate, that the $2P$ would double in fewer than n years. Actually it will take the same number of years for P and $2P$ to double.

Suppose a person rows two miles upstream and then two miles down a river that has a current of three miles per hour. Let us assume that the person can row five miles per hour in still water. How long should the entire trip take? Intuition suggests that the current will help the person as much on the way downstream as it will hinder on the way upstream. Hence the person will row four miles at five miles per hour, and the total time will be four-fifths of an hour. Actually the intuition is wrong, and the total time is one and a quarter hours.

Suppose one adds a quart of vermouth to a quart of gin to make a somewhat delectable martini. One should expect to obtain two quarts of martini. The correct answer, which certainly will elude the

intuition, is one and nine-tenths quarts. Likewise, five pints of water and seven pints of alcohol do not make twelve pints of the mixture. In both cases molecules combine.

Let us now consider the problem of time. We can speak of the next second after a given second. A second is merely a duration of time. Intuition suggests that there is a next instant to a given instant. By instant we mean *no* duration of time, as for example the instant when a clock strikes one. But let us consider the paradox first presented by Zeno of Elea (fifth century B.C.). An arrow in flight is at one position at any one instant and at another position at the next one. How did the arrow have time to get to the next position?

Let us consider next a related problem of time. A clock strikes six in five seconds. How long should it take to strike twelve? Seemingly ten seconds. However, there are five intervals between the six strokes and eleven intervals between the twelve strokes. Hence the correct answer is eleven, not ten.

Let us consider a few more examples of intuition's failure. Consider two rectangles with the same perimeter. Must they have the same area? It would seem so. However, a little arithmetic soon tells us that this need not be so. We should then ask, of all rectangles with the same perimeter, which has the largest area? After all, if fencing is to be used to enclose the rectangle and if the area is to be used for planting, then the rectangle with the largest area is most desirable. The answer is a square.

A related problem asks us to consider two boxes of the same volume. Must the total area of the six sides of each be the same? Let us suppose that each box has a volume of 100 cubic feet. One can have dimensions of 50 by 1 by 2 feet, and the other 5 by 5 by 4 feet. The surface areas in this case are 204 and 130 square feet, respectively. Clearly, the difference is striking.

Another example of where intuition fails involves a young man who has a choice between two jobs. Each offers a starting salary of $1800 per year, but the first one would lead to an annual raise at the rate of $200, whereas the second would lead to a semiannual raise at the rate of $50. Which job is preferable? One would think that the answer is obvious. A raise of $200 per year seems better than one that apparently would amount to only $100 per year. But let us do a little arithmetic and consider what each job offers *during* successive *six-month* periods. The first job will pay 900, 900, 1000, 1000, 1100, 1100, 1200, 1200. . . . The second job, which provides a semiannual increase of $50, will pay 900, 950, 1000, 1050, 1100, 1150, 1200, 1250. . . .

It is clear from a comparison of these two sets of salaries that the second job brings a better return during the second half of each year and does as well as the first job during the first half. The second job is the better one. With the arithmetic before us it is possible to see more readily why the second job is better. The semiannual increase of $50 means that the salary will be higher at the rate of $50 for six months or at the rate of $100 for the year, because the recipient will get $50 more for each of the six-month periods. Hence two such increases per year amount to an increase at the rate of $200 per year. Thus far the two jobs seem to be equally good. However, on the second job the increases start after the first six months, whereas on the first job they do not start until after one year has elapsed. Hence the second job will pay more during the latter six months of each year.

Let us consider another simple problem. Suppose that a merchant sells his apples at two for five cents and his oranges at three for five cents. Being somewhat annoyed with having to do considerable arithmetic on each sale, the merchant decides to commingle apples and oranges and to sell any five pieces of fruit for ten cents. This move seems reasonable, because if he sells two apples and three oranges he sells five pieces of fruit and receives ten cents. Now he can charge two cents apiece, and his arithmetic on each sale is simple.

The dealer is cheating himself. Just to check quickly, we shall assume that he has one dozen apples and one dozen oranges for sale. If he sells apples normally at two for five cents, he receives thirty cents for the dozen apples. If he sells oranges at three for five cents, he receives twenty cents for the dozen oranges. His total receipts are then fifty cents. However, if he sells the twenty-four pieces at five for ten cents he will receive two cents per article or forty-eight cents.

The loss is due to poor reasoning on the part of the dealer. He assumed that the average price of the apples and oranges should be two cents each; however the average price per apple is two and one-half cents and the average price per orange one and two-thirds cents. The average price of two such items is two and one-twelfth cents per article and not two cents.

Next let us consider another common faulty intuition. Suppose we have a circular garden with a radius of 10 feet. We wish to protect the garden by a fence that is to be at each point 1 foot beyond the boundary of the garden. How much longer is the fence than the circumference of the garden itself? The answer is readily obtained. The circumference of the garden is given by a formula of geometry; this says that the circumference is 2π times the radius, π being the symbol for a number that is approximately 22/7. Hence the circumference of the garden is $2\pi \times 10$. The condition that the fence be 1 foot beyond the garden means that the radius of the circular fence is to be 11 feet. Hence the length of the fence is $2\pi \times 11$. The difference in these two circumferences is $22\pi - 20\pi$ or 2π. Therefore, the fence should be 2π feet longer than the circumference of the garden. There is nothing remarkable thus far.

We now consider a related problem. Suppose we were to build a roadway around the Earth—a trivial task for modern engineers—and the height of the roadway were to be 1 foot above the surface of the Earth all the way around. How much longer than the circumference of the Earth would the roadway be? Before calculating this quantity let us use our intuition at least to estimate it. The radius of the Earth is about 4000 miles or 21,120,000 feet. Since this radius is roughly two million times that of the garden we considered, one might expect that the *additional* length of the roadway should be about two million times the additional length of fence required to enclose the garden. The latter quantity was just 2π feet. Hence an intuitive argument for the additional length of roadway would seemingly lead to the figure of $2{,}000{,}000 \times 2\pi$ feet. Whether or not you would agree to this argument, you would almost certainly estimate that the length of the roadway would be very much greater than the circumference of the Earth.

A little mathematics tells the story. To avoid calculation with large numbers, let us denote the radius of the Earth by r. The circumference of the Earth is then $2\pi r$. The circumference or length of the roadway is $2\pi (r + 1)$. But the latter equals $2\pi r + 2\pi$. Hence the difference between the length of the roadway and the circumference of the Earth is just 2π feet, precisely the same figure that we obtained for the difference between the length of fence and the circumference of the garden, even though the roadway encircles an enormous Earth, whereas the fence encircles a small garden. In fact, the mathematics tells us even more. Regardless of what the value of r is, the difference, $2\pi (r + 1) - 2\pi r$, is always 2π, and this means that the circumference of the outer circle, if it is at each point 1 foot away from the inner circle, will always be just 2π feet longer than the circumference of the inner circle.

Intuition fails us in many other situations. A man some distance away from a tree notes that an apple is about to fall and wishes to hit the apple with a rifle bullet. He knows that by the time the bullet reaches the apple, it will have fallen some distance. Should he then aim at a point somewhat below the apple so that his bullet will hit it? No.

He should aim and fire at the apple. Both will fall the same distance downward during the time that the bullet is in flight.

As a final example of where intuition is likely to fail let us suppose there are 136 entrants in a tennis tournament and the director wishes to schedule a minimum number of matches to select the winner. How many need he schedule? Intuition seems helpless. The answer is 135, because each contestant must be defeated once, and once defeated, is eliminated.

Why are we subject to illusions of the senses and to false intuitions? Examination of the physiology of the various sensory organs could explain the sensory illusions, but for our purposes all we need is to recognize that the human sensory organs and the human brain are involved. With regard to intuition, it is actually a combination of experience, sense impressions, and crude guessing; at best one could say intuition is distilled experience. Subsequent analyses or experiments confirm or discredit it. Intuitions have been characterized as no more than force of habit rooted in psychological inertia.

When we speak of what is certain perceptually, we presuppose a separation between the perception and the perceiver. But this is impossible, because there can be no perception without a perceiver. What, then, is objective? We perhaps naively assume that what all perceivers agree on is objective. There are a sun and a moon. The sun is yellow and the moon is blue.

Helmholtz said in the *Handbook of Physiological Optics* (1896):

> It is easy to see that all the properties we ascribe to them [objects of the external world] signify only the effects they produce either on our senses or other external objects. Color, sound, taste, smell, temperature, smoothness, solidity belong to the first class; they signify effects on our sense organs. The chemical properties are likewise related to reactions, i.e., effects which the natural body in question exerts on others. It is thus with the other physical properties of bodies, the optical, the electric, the magnetic. . . . From this it follows that in fact, the properties of objects in nature do not signify, in spite of their name, anything proper to the objects in and for themselves but always a relation to a second body (including our sense organs).

What is our recourse to countering illusions and erroneous intuitions? The most effective answer is the use of mathematics. Just how effective the subject is remains to be seen. Our chief concern will be to show that there are phenomena in our physical world that are as real as any we perceive through our senses but that are extrasensory or not at all perceptible and, in fact, that in our present culture we utilize and

rely on these extrasensory real phenomena, at least as much and perhaps even more than we rely on our sense perceptions.

This is not to say that mathematics does not utilize perceptions and intuitions as suggestions for its own development. However, mathematics surpasses these suggestions much as a diamond surpasses a piece of glass, and what mathematics reveals about our physical world is far more astonishing than the spectacle of the heavens.

II

The Rise and Role of Mathematics

In every specific natural science there can be found only so much science proper as there is mathematics present in it. Kant

The gods have not revealed all things from the beginning, but men seek and so find out better in time. Xenophanes

For the apparel oft proclaims the man. Shakespeare

Although the information obtained through our senses has been carefully observed, measured, and checked by experimentation, and although we now can utilize such aids as the telescope and the microscope, surveying instruments, and remarkably accurate measuring devices, the knowledge thus acquired is still limited and only approximately accurate. We know more about the number of planets, the presence of satellites of several planets, dark spots on the sun, and the use of the compass to navigate the seas. Yet all of these gains in knowledge are insignificant in comparison with the variety and importance of the phenomena we need to and wish to study.

The crucial, powerful, and decisive step that has increased and enhanced our knowledge of the physical world is the employment of mathematics. The role of this tool is so far superior to the means just described that it can be labeled superlative and even miraculous. Not only does it correct and increase our knowledge of phenomena that are perceptible, it also reveals vital phenomena that are not at all perceptible but that are as real in their effects as touching a hot stove. There are physical "ghosts" whose presence in our daily lives cannot be doubted. How their existence was disclosed will be our next focus of discussion.

To those of us who were educated in Western Europe and the Americas, the nature and mundane uses of mathematics are familiar and accepted as almost commonplace. Even the civilizations we credit as originating Western European mathematics, namely, the Babylonian and Egyptian civilizations, produced from about 3000 B.C. a collection of useful but disconnected rules and formulas to solve such practical problems as humans encounter in their everyday life. These peoples did not recognize the power of mathematics to extend their knowledge of nature beyond what the senses reveal. Their mathematics may be regarded as the alchemy that preceded chemistry.

Mathematics, as a logical development and a tool to learn more about nature, is a creation of the Greeks that they began to take seriously about 600 B.C. There are no surviving documents of the sixth and fifth centuries B.C. that can tell us how or why the Greeks came to this new concept and role of mathematics. Instead, we have the speculations of historians, one of whom states that the Greeks found contradictory results on the area of a circle in the works of Babylonians and Egyptians and so had to decide which was correct. There were similar disagreements on other topics. Another explanation cites the philosophical interests of the Greeks, but this suggestion raises more questions than it answers. Still another sees deductive mathematics originating from Aristotelian logic, which arose from disputation on political and social issues. But Greek mathematics precedes Aristotle.

Perhaps all one can say is that from the sixth century B.C. onward the Greeks had a vision, the substance of which was that nature is rationally designed and that all natural phenomena follow a precise and unvarying plan, indeed a mathematical plan. The human mind has superb power, and if this power is applied to the study of nature, the rational, mathematical pattern can be discerned and rendered intelligible.

In any case the Greeks became the first people with the audacity and the genius to give reasoned explanations of natural phenomena. The Greek urge to understand had the excitement of a quest and an exploration. While they explored they made maps, such as Euclidean geometry, so that others might find their way quickly to the frontiers and help conquer new regions.

We are on somewhat safer historical grounds when we cite that Thales (*c.* 640–*c.* 546 B.C.), who lived in the Greek city of Miletus in Asia Minor, proved several theorems of Euclidean geometry. There are no documents of his time, and the belief that he proved theorems by logical means is somewhat dubious; yet it is certain that he and his contemporaries in Asia Minor speculated on the design of nature.

We are more certain that with the advent of the Pythagoreans, a mystical and religious order of the sixth century B.C., the program of determining the rational design of nature enlisted the aid of mathematics. The Pythagoreans were struck by the fact that phenomena that are physically very diverse exhibit identical mathematical properties. The moon and a rubber ball share the same shape and many other properties common to spheres. Was it not apparent, then, that mathematical relations underlie diversity and must be the essence of phenomena?

To be specific, the Pythagoreans found this essence in number and in numerical relations. Number was the first principle in the description of nature, and it was the matter and form of the universe. The Pythagoreans are reported to have believed that "all things are numbers." This belief makes more sense when we take into account that the Pythagoreans visualized numbers as dots (which could have meant particles to them) and that they arranged dots in patterns, each of which could be taken to represent a real object. Thus the collections

were called triangular and square numbers, and may have been regarded as representing triangular and square objects. There is no doubt that as the Pythagoreans developed and refined their own doctrines they began to understand numbers as abstract concepts and physical objects as concrete realizations.

The Pythagoreans are credited with the reduction of music to simple relationships among numbers when they discovered two facts: first, the sound caused by a plucked string depends on the length of the string; and second, harmonious sounds are given off by strings whose lengths can be expressed as ratios of whole numbers. For example, a harmonious sound is produced by plucking two equally taut strings, one twice as long as the other. The musical interval between the two notes is now called an octave. Another harmonious combination is formed by plucking two strings whose lengths are in the ratio of three to two; in this case the shorter one gives forth a note called the fifth above that given off by the longer one. In fact, the relative lengths in

every harmonious combination of plucked strings can be expressed as ratios of whole numbers.

The Pythagoreans also "reduced" the motions of the planets to numerical relations. They believed that bodies moving in space produce sounds and that a body moving rapidly gives forth a higher note than one moving more slowly. Perhaps these ideas were suggested by the swishing sound of an object whirled on the end of a string. According to Pythagorean astronomy, the greater the distance of a planet from the Earth the more rapidly it moved. Hence the sounds produced by the planets varied with their distance from the Earth, and these sounds all harmonized. But this "music of the spheres," like all harmony, could be reduced to mere numerical relationships, and hence so could the motions of the planets.

Other features of nature were "reduced" to number. The numbers 1, 2, 3, 4, the *tetractys*, were especially valued. In fact, the Pythagorean oath is reported to be: "I swear in the name of the Tetractys which has been bestowed on our soul. The source and roots of the overflowing nature are contained in it." Nature was composed of "fournesses," such as the four geometrical elements (point, line, surface, and solid), and the four material elements Plato later emphasized (earth, air, fire, and water).

The four numbers of the *tetractys* added up to ten, and so ten was the ideal number and represented the universe. Because ten was ideal there must be ten bodies in the heavens. To specify the required number of bodies the Pythagoreans introduced a central fire around which the Earth, sun, moon, and the five planets then known revolved, and a counter-earth on the opposite side of the central fire. Neither this central fire nor the counter-earth is visible because the area of the Earth on which we live faces away from them. In this way the Pythagoreans built an astronomical theory based on numerical relationships.

With these examples we can make sense of the statement attributed to Philolaus, a famous fifth-century B.C. Pythagorean:

> Were it not for number and its nature nothing that exists would be clear to anybody either in itself or in its relation to other things. . . . You can observe the power of number exercising itself not only in the affairs of demons and gods but in all the acts and thoughts of men, in all handicrafts and music.

The natural philosophy of the Pythagoreans is hardly substantial. Moreover, the Pythagoreans did not develop any one branch of physical science very far. Justifiably, one could call their theories superficial. Whether by luck or by intuitive genius, however, the Pythagoreans did advance two doctrines that proved later to be all-important: the first is that nature is built according to mathematical principles, and the second, that number relationships underlie, unify, and reveal the order in nature.

The atomists Leucippus (*c.* 440 B.C.) and Democritus (*c.* 460–*c.* 370 B.C.) also advanced the importance of mathematics. They believed that all matter consists of atoms that differ in position, size, and shape. These were physically real properties of the atoms. All other properties such as taste, heat, and color were not in the atoms but in the effect of the atoms on the perceiver. This sensuous knowledge was unreliable because it varied with the perceiver. Like the Pythagoreans, the atomists asserted that the reality underlying the constantly changing features of the physical world was expressible in terms of mathematics. Thus, the happenings in this world were strictly determined by mathematical laws.

The Greek who most effectively promoted the mathematical investigation of nature was Plato (427–347 B.C.). Plato took over some Pythagorean doctrines but was a master in his own right who dominated Greek thought in the momentous fourth century B.C. He was the founder of the Academy in Athens, a center that attracted the leading thinkers of his day and that endured for nine hundred years. His views are clearly expressed in his dialogue *Philebus.* We have already noted (see Historical Overview) that the real world according to Plato was designed mathematically. What we perceive through our senses is an imperfect representation of the real world. The reality and intelligibility of the physical world could be comprehended only through mathematics, for "God eternally geometrizes." Plato went further than most Pythagoreans in that he wished not merely to understand nature through mathematics but to transcend nature to comprehend the ideal, mathematically organized world that he believed to be the true reality. The sensory, the impermanent, and the imperfect were to be replaced by the abstract, eternal, and perfect. He hoped that a few penetrating observations of the physical world would suggest basic truths that could then be developed through reason; at this point, he could dispense with further observation. And from here on, nature would be replaced entirely by mathematics. Indeed, he criticized the Pythagoreans because they investigated the numbers of the harmonies that are heard but never reached the natural harmonies of numbers themselves. For Plato, mathematics was not only the mediator between the ideas and the things of sense; the mathematical order was the true account of the nature of reality. Plato also laid down the principles of the axiomatic-deductive method, which we shall discuss shortly. He saw this method as the ideal way of systematizing knowledge and arriving at new knowledge.

The pursuit of mathematics to study and obtain true knowledge of our physical world was urged also by the leading successor of Plato. Although Aristotle and his followers differed somewhat from the Platonists concerning the relationship of mathematics to the real world, this school also expounded and advocated the mathematical design of nature. Aristotle affirmed that the abstractions of mathematics were derived from the material world; however, there are no passages in his writings that advocate mathematics as a correction or extension of sensory knowledge. He did believe that the motions of the heavenly bodies were mathematically designed but, basically, that mathematical laws were merely a description of events. For Aristotle the final cause or objective of events, the teleological doctrine, was most important.

When Alexander the Great (356–323 B.C.) set out to conquer the world, he transferred the center of the Greek world from Athens to a city of Egypt, which he modestly named Alexandria. It was in Alexandria that Euclid (*c.* 300 B.C.) wrote the first memorable document on mathematical knowledge, the classic *Elements.* Here proof makes its first known appearance. Euclid also wrote tracts on mechanics, optics, and music in which mathematics was the core; mathematics was the ideal version of what the known physical world contained. Some of his theorems indeed offered new knowledge of geometrical figures and of properties of the whole numbers. However, because we have no original manuscripts by Euclid we do not know whether new knowledge was his objective or whether he was concerned with the reliability of sensory knowledge. In any case he led the way for other creators of mathematics.

The Greeks of the Alexandrian period (*c.* 300 B.C.–A.D. 600) extended mathematics almost immeasurably. For present purposes we need only note the major work by Apollonius (*c.* 262–*c.* 190 B.C.), *Conic Sections;* a variety of first-class works by Archimedes (*c.* 287–212 B.C.) on many areas of mathematics and mechanics; the work on trigonometry by Hipparchus, Menelaus, and Ptolemy (*c.* A.D. 85–*c.* 165); and, late in the period (*c.* A.D. 250), the arithmetical work of Diophantus. All of these works, like Euclid's, gave ideal versions of objects, relationships, and phenomena of the physical world and extended our knowledge.

The Greek civilization was destroyed by the conquests of the Romans and Mohammedans, and with its demise Europe entered the Middle Ages, which endured for a thousand years from about A.D. 500 to 1500. This culture was dominated by the Catholic church, which subordinated life on Earth to preparation for an afterlife in heaven. Consequently, the study of nature by any means, mathematical or otherwise, was deprecated. Nevertheless, a few individuals and groups (Robert Grosseteste, Roger Bacon, John Peckham, the Mertonians at Oxford—among whose members were William of Ockham, Thomas Bradwardine, Abelard of Bath, Thierry of Chartres, and William of Conches) did make some efforts to continue mathematical and physical investigation. In particular they subscribed to mathematics as the veridical account of physical phenomena, and some, notably Abelard and Thierry, insisted also on experimental techniques. All of these thinkers believed that the universe was basically rational and that mathematical reasoning could produce knowledge about it. Nor should we overlook during this medieval period the contributions of the Hindus and the Arabs, which were gradually absorbed into the body of mathematics.

The modern period, our main concern, may be thought to begin about 1500. The sixteenth century in particular is often distinguished as the Renaissance, the rebirth of Greek thought. Just how Greek manuscripts reached Italy, the center of the Renaissance, is irrelevant to our account. It may suffice to say that the Greek ideas fascinated the Europeans.

The Europeans generally did not respond immediately to the new forces and influences. During the period often labeled "humanistic," the study of Greek works was far more characteristic than the active pursuit of Greek objectives, but by about A.D. 1500 European minds, infused with Greek goals—that is, the application of reason to the study of nature and the search for the underlying mathematical design—began to act. However, they faced a serious problem in that the Greek goals were in conflict with the prevailing culture. Whereas the Greeks believed in the mathematical design of nature, with nature conforming invariably and unalterably to some ideal plan, late medieval thinkers ascribed all plan and action to the Christian God. He was the designer and creator, and all the actions of nature followed the plan laid down by this agency. The universe was the handiwork of God and subject to His will. The mathematicians and scientists of the Renaissance and several succeeding centuries were orthodox Christians and so accepted this doctrine. But Catholic teachings by no means included

the Greek doctrine of the mathematical design of nature. How, then, was the attempt to understand God's universe to be reconciled with the search for the mathematical laws of nature? The answer was to add a new doctrine—that the Christian God had designed the universe mathematically. Thus, the Catholic doctrine postulating the supreme importance of seeking to understand God's will and His creations took the form of a search for God's mathematical design of nature. Indeed, the work of sixteenth-, seventeenth-, and most eighteenth-century mathematicians was, as we shall soon see more clearly, a religious quest. The search for the mathematical laws of nature was an act of devotion that would reveal the glory and grandeur of His handiwork.

Mathematical knowledge, the truth about God's design of the universe, thus became as sacrosanct as any line of Scripture. Humans could not hope to perceive the divine plan as clearly as God Himself understood it, but humans could with humility and modesty seek at least to approach the mind of God and so understand God's world.

One can go further and assert that these mathematicians were sure of the existence of mathematical laws underlying natural phenomena and persisted in the search for them because they were convinced a priori that God had incorporated them into the construction of the universe. Each discovery of a law of nature was hailed as evidence of God's brilliance rather than that of the investigator. The beliefs and attitudes of the mathematicians and scientists swept Renaissance Europe. The recently discovered Greek works confronted a deeply devout Christian world, and the intellectual leaders born in one word and attracted to the other fused the doctrines of both.

Alongside this new intellectual fervor, another doctrine was gaining support—the idea of "back to nature." Every variety of scientist abandoned endless rationalizing on the basis of dogmatic principles, vague in meaning and unrelated to experience, and turned to nature herself as the true source of knowledge. Certainly by 1600 the Europeans were motivated to undertake what has often been described as the Scientific Revolution. Several happenings motivated or accelerated this movement: geographical explorations disclosed new lands and new peoples; the invention of the telescope and microscope revealed new phenomena; the compass aided navigation; the heliocentric theory introduced by Copernicus (see Chapter IV) stimulated new thoughts about our planetary system; and the Protestant Revolution challenged Catholic doctrines. Mathematics soon resumed its major role as the key to nature.

This brief sketch of the historical background of modern European mathematics is intended primarily to indicate that mathematics and its uses in the investigation of nature, which will be our main concern in succeeding chapters, did not originate as a bolt from the blue. However, our concern will not be the elementary mathematics that provided the tools to correct and extend our knowledge of commonly perceptible phenomena but, instead, what mathematics has achieved in revealing and describing phenomena that either are not readily accessible or are not at all so through the senses. For our purposes we need not pursue and master the techniques of mathematics, but it is essential to understand how mathematics enables us to represent physical phenomena and to arrive at knowledge about these phenomena.

What are the essential elements of the mathematical method? The first is the introduction of basic concepts. Some, such as point, line, and whole number, are suggested directly by material or physical objects. Beyond the elementary concepts, mathematics has in fact become dominated by concepts derived from the recesses of the human mind. To cite a few examples of such concepts: negative numbers, letters standing for classes of numbers, complex numbers, functions, all sorts of curves, infinite series, concepts of the calculus, differential equations, matrices and groups, and higher dimensional spaces.

Some of the above concepts lack entirely an intuitive meaning. Others do have some intuitive basis in physical phenomena as, for example, the *derivative* or instantaneous rate of change. However, although it is related to the physical phenomenon of velocity, the derivative is far more an intellectual construct and is qualitatively an entirely different sort of contribution from that of the mathematical triangle.

Throughout the history of mathematics new concepts were viewed with suspicion at the outset. Even the notion of negative numbers was originally rejected by serious mathematicians. However, each new concept was grudgingly accepted as its usefulness in application became evident.

A second essential feature of mathematics is abstraction. Speaking of geometricians, Plato said in the *Republic:*

> Do you not know also that although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of absolute square and the absolute diameter . . . they are really seeking to behold the things themselves which can be seen only with the eye of the mind?

If mathematics is to be powerful, it must embrace in one abstract concept the essential features of all the physical manifestations of that concept. Thus, the mathematical straight line must embrace stretched strings, rulers' edges, boundaries of fields, and the paths of light rays.

That the concepts are abstractions is exemplied in the most elementary concept, *number.* Failure to recognize this can lead to confusion. A simple situation can be used to make the point. A man goes into a shoestore and buys three pairs of shoes at $20 a pair. The salesperson says that three pairs of shoes at $20 a pair cost $60 and he expects the customer to hand him $60. But the customer instead replies that three pairs of shoes at $20 a pair is not $60 but sixty pairs of shoes, and he asks the salesperson for the sixty pairs. Is the customer right? As right as the salesperson. If pairs of shoes times dollars can yield dollars, then why cannot the same product yield pairs of shoes? The answer is, of course, that we do not multiply shoes by dollars. We abstract the numbers three and twenty from the physical situation, multiply to obtain sixty and then interpret the result to suit the physical situation.

Another feature of mathematics is idealization. A mathematician idealizes by deliberately ignoring the thickness of a chalk mark in treating straight lines or by regarding the Earth as a perfect sphere in some problems. Idealization per se is not a serious departure from reality, but it does raise the question in any application to reality whether the real particle or path under study is close enough to its idealization.

The most striking feature of mathematics is the method of reasoning it employs. The basis is a set of axioms and the application of deductive reasoning to these axioms. The word *axiom* comes from the Greek, meaning "to think worthy." The Greeks introduced the notion of axioms—truths so self-evident that no one could doubt them. Plato's theory of *anamnesis* stated that humans had a prior experience of truth as souls in an objective world of truths and that the axioms of geometry represented the recollection of previously known truths. Aristotle maintained in *Posterior Analytics* that the axioms are known to be true by our infallible intuition. Moreover, we must have these truths on which to base our reasoning. If, instead, reasoning were to use some facts not known to be truths, further reasoning would be needed to establish these facts, and this process would have to be repeated endlessly. Aristotle also pointed out that some concepts must remain undefined or else there would be no starting point. Today such terms as point and straight line are undefined; their meaning and properties depend on the axioms that prescribe their properties.

Just as many of the concepts with which mathematics deals are invented by human minds, so the axioms about these concepts are invented to suit what the concepts are intended to reveal about reality. Thus, axioms for negative and complex numbers must necessarily be different from those for positive numbers, or at least the latter must be extended to include negative and complex numbers. The subtleties in the newer concepts are far greater, and the correct axiomatic bases for some branches of mathematics were achieved long after the branch was established.

Beyond mathematical axioms, some physical knowledge must enter into major contributions of mathematics to our physical world. This may take the form of physical axioms such as Newton's laws of motion, generalizations of experimental observations, or sheer intuition. These physical assumptions are formulated in the language of mathematics and so permit the axioms and theorems of mathematics to be applied to them.

However basic the concepts and axioms, it is the *deductions* from the axioms that allow us to acquire totally new knowledge to correct our sense perceptions. Of the many types of reasoning—for example, inductive, analogical, and deductive—only deductive guarantees the correctness of the conclusion. To conclude that all apples are red because 1000 apples are found to be red is inductive reasoning, therefore not reliable. Similarly, the argument that John should be able to graduate from college because his identical twin who inherited the same faculties did so, is reasoning by analogy, and is certainly not reliable. Deductive reasoning, on the other hand, although it can take many forms, does guarantee the conclusion. Thus, if one grants that all men are mortal and Socrates is a man, one must accept that Socrates is mortal. The principle of logic involved here is one form of what Aristotle called syllogistic reasoning. Among other laws of deductive reasoning Aristotle included the law of contradiction (a proposition cannot be both true and false) and the law of excluded middle (a proposition must be either true or false).

He and the world at large accepted unquestioningly that these deductive principles when applied to any premises yielded conclusions as reliable as the premises. Hence, if the premises were truths, so would be the conclusions. It is worthy of note that Aristotle abstracted the principles of deductive logic from the reasoning already practiced by mathematicians. Deductive logic is, in effect, the child of mathematics.

It is important to appreciate how radical the insistence on deductive proof is. We can test as many even numbers as we wish and find

that each is a sum of two prime numbers. However, we can not state that this result is a mathematical theorem because it was not obtained by a deductive proof. Similarly, suppose a scientist were to measure the sum of the angles of 100 different triangles in different locations and of different size and shape, and find that sum to be 180 degrees to within the limits of experimental accuracy. Surely this scientist would conclude that the sum of the angles of any triangle is 180 degrees. However, not only were the measurements approximate, there remained the question of whether some triangular shape not measured would produce a markedly different result. The scientist's inductive proof is not mathematically acceptable. The mathematician, on the other hand, starts with facts or axioms that seem to be reliable. Who can doubt that if equals be added to equals, the sums are equal? By means of such indubitable axioms one can prove deductively that the sum of the angles of *any* triangle is 180 degrees.

The deductive process we have described uses logic to justify the reasoning. What has been employed practically up to modern times is what is called Aristotelian logic. We may ask why the conclusions derived by this application of logic should apply to nature. Why should theorems deduced by human minds sitting in cloistered rooms be as applicable to the real world as the axioms that are themselves in many cases also suggested only by human minds? We shall return to the question of why mathematics works in Chapter XII.

We have yet to cite another important feature of mathematics—the use of symbolism. Although a page of mathematical symbols can hardly be described as appealing, there is no question that without symbolism mathematicians would be lost in a wilderness of words. All of us use symbolism in a host of common abbreviations. We use N.Y. to mean New York, for example, and although the meaning of such symbols must be learned, there is no question that the brevity of symbolism permits comprehension, whereas a verbal expression would overburden the mind.

We can sum up the means by which mathematics derives facts about our physical world by saying that it builds models for classes of real phenomena. Concepts, usually idealized (whether drawn from observation of nature or supplied by human minds); axioms, which may also be suggested by physical facts or by humans; and the processes of idealization, generalization, and abstraction, as well as intuition, are all utilized in the building of models. Proof, of course, cements the components of a model. The most familiar model is Euclidean geometry, but we shall examine many more sophisticated and more ingenious models that tell us far more about far less obvious phenomena than Euclidean geometry.

Our goal, then, is to see how firmly mathematics enters the modern world, not just as the method of correcting the imperfections of the senses but more especially as a method of extending the knowledge humans can acquire about their world. As Hamlet put it, "There are more things in Heaven and earth, Horatio, than are dreamt of in your philosophy." We must go beyond perceptual knowledge. The essence of mathematics, as opposed to sense perceptions, is that it draws on the human mind and human reasoning to produce knowledge about our physical world that the average human being, even in Western culture, believes is acquired entirely by the use of sense perceptions.

Alfred North Whitehead, in his *Science and the Modern World*, has emphasized the importance of mathematics in the exploration of our physical world.

> Nothing is more impressive than the fact that as mathematics withdrew increasingly into the upper regions of ever greater abstract thought, it returned back to earth with a corresponding growth of importance for the analysis of concrete fact. . . . The paradox is now fully established that the utmost abstractions are the true weapon with which to control our thought of concrete fact.

And as David Hilbert, the foremost twentieth-century mathematician, remarked, physics is now too important to be left to the physicists.