THINK MATHS

IS MATHEMATICS THE GRAND DESIGN FOR THE UNIVERSE, OR MERELY A FIGMENT OF THE HUMAN IMAGINATION?

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WHERE does mathematics come from? Is it already out there, waiting for us to discover it, or do we make it all up as we go along? Plato held that mathematical concepts actually exist in some weird kind of ideal reality just off the edge of the Universe. A circle is not just an idea, it is an ideal. We imperfect creatures may aspire to that ideal, but we can never achieve it, if only because pencil points are too thick. But there are those who say that mathematics exists only in the mind of the beholder. It does not have any existence independent of human thought, any more than language, music or the rules of football do.

Nature's Patterns

So who is right? Well, there is much that is attractive in the Platonist point of view. It's tempting to see our everyday world as a pale shadow of a more perfect, ordered, mathematically exact one. For one thing, mathematical patterns permeate all areas of science. Moreover, they have a universal feel to them, rather as though God thumbed His way through some kind of mathematical wallpaper catalogue when He was trying to work out how to decorate His Universe. Not only that: the deity's pattern catalogue is remarkably versatile, with the same patterns being used in many different guises. For example, the ripples on the surface of sand dunes are pretty much identical to the wave patterns in liquid crystals. Raindrops and planets are both spherical. Rainbows and ripples on a pond are circular. Honeycomb patterns are used by bees to store honey (and to pigeonhole grubs for safekeeping), and they can also be found in the geographical distribution of territorial fish, the frozen magma of the Giant's Causeway, and rock piles created by convection currents in shallow lakes. Spirals can be seen in water running out of a bath and in the Andromeda Galaxy. Frothy bubbles occur in a washing-up bowl and the arrangement of galaxies.

With this kind of ubiquitous occurrence of the same mathematical patterns, it is no wonder that physical scientists get carried away and declare them to lie at the very basis of space, time and matter. Eugene Wigner expressed surprise at the "unreasonable effectiveness" of mathematics as a method for understanding the Universe. Many philosophers and scientists have seen mathematics as the basis of the Universe. Plato wrote that "God ever geometrises". The physicist James Jeans declared that God was a mathematician. Paul Dirac, one of the inventors of quantum mechanics, went further, opining that He was a pure mathematician. In the past few years Edward Fredkin has argued that the Universe is made from information, the raw material of mathematics.

This is powerful, heady stuff, and it is highly appealing to mathematicians. However, it is equally conceivable that all of this apparently fundamental mathematics is in the eye of the beholder, or more accurately, in the beholder's mind.

We human beings do not experience the Universe raw, but through our senses, and we interpret the results using our minds. So to what extent are we mentally selecting particular kinds of experience and deeming them to be important, rather than picking up things that really are important in the workings of the Universe? Is mathematics invented or discovered?

If pushed, I would say that it is a bit of both because neither word adequately describes the process. Moreover, they are not alternatives, they are not opposites, and they do not exhaust the possibilities. They are not even particularly appropriate. We use "discover" for finding things that already exist in the physical world. Columbus discovered America—it was already there, but neither he nor anyone else where he came from knew it was—and David Livingston discovered the Victoria Falls. The word "invention" means bringing into existence something that was not previously there. Edison invented electric light, Bell invented the telephone.

However, when Columbus landed in America he was actually trying to invent a new trade route to India. And Livingston's discovery came as no great surprise to the local inhabitants who saw the Victoria Falls every day. Edison would have felt as if he had invented the idea of electric lighting, but then spent many years trying to discover how to make it a reality. So invention and discovery both happen within a particular context—people becoming aware that there is something new in their world.

It is the same with mathematics. What to the outside world looks like invention often feels more like discovery to insiders. The distinction is made all the more tricky because mathematical objects lead a virtual existence, not a real one: they reside in minds, not embodied in any kind of hardware. But unlike, say, poetry, that virtual world obeys rigid rules, and those rules are pretty much the same in every mathematical mind.

In a way, the world of mathematical ideas is a kind of virtual collective comparable to Jung's famous "collective unconscious"—the idea that all human minds have access to vast, evolutionarily ancient, subconscious structures and processes that govern much of our behaviour. But in what sense are they "collective"? A crucial distinction has to be made here between a single unconscious entity, into which we all dip, and a large number of distinct but very similar unconsciousnesses, one for each of us. It is the difference between a community with a single municipal swimming pool, and one in which every back garden has its own pool.

From the point of view of specific action, the distinction is not terribly important: you can discuss the problems of keeping leaves out of "the pool" with your neighbor without ever making it clear whether you think of it as a single common pool or a typical representative of the individual pools that everybody has. But if you want to understand what's going on in general, then it does make a difference. The notion of a single unconscious mind for all of humanity is a mystical and rather silly concept that leads in the direction of telepathy. A collection of more or less identical individual subconsciousnesses, rendered similar by their common social context, is considerably more prosaic but a great deal more sensible.

The same point lies at the heart of how I think we should view mathematics. Because we have a single word for the virtual collective it is tempting to think of it as a single thing—like Jung's mystical telepathic unconscious—into which all mathematicians dip. This is a difficult concept to capture. Where is that thing? What is it made of? How does it grow? Instead, it is better to think of mathematics as being distributed throughout the minds of the world's mathematicians. Each has his or her own mathematics inside his or her head. Moreover, those individual systems are extremely similar to each other, much more so than Jungian subconsciousnesses. Not in the sense that each head contains the whole of mathematics. Mine contains dynamical systems, yours contains analysis, and hers algebra, say. But all three are logically consistent with each other, because of how mathematicians are trained, and how they communicate their ideas. If what is in my head is not consistent with what is in yours, then one of us has got it wrong and we will argue until it becomes clear to us both who it is.

Baking Bread

Most areas of human activity are structured in this way. So the difficult questions of existence and discovery versus invention are not confined to mathematics. Take medicine, for example. What is medicine? Where does it live? Is it invented or discovered? Now replace medicine by plumbing, ballet, football, language or cycling, and it is clear just how widespread the structure is, and why the question doesn't make a great deal of sense in any area of human activity. What goes on is neither invention nor discovery, but a complex context-dependent mix of both.

When it comes to mathematics, sometimes it really does feel like discovery. When you are carrying out mathematical research in a previously defined area it feels like discovery because there is no choice about what the answer is. But when you are trying to formalise an elusive idea or find a new method, it feels more like invention: you are floundering around, trying all sorts of harebrained ideas, and you simply do not know where it will all lead. The more established an area of mathematics becomes, the more strongly it feels as if there is some kind of fixed logical landscape, which you merely explore. Once you've made a few assumptions (axioms), then everything that follows from them is predetermined. But this account misses out the most crucial features: significance, simplicity, elegance, how compelling the argument is, all things that give the landscape its character.

But if mathematics resides in mathematicians' heads, why is it so "unreasonably effective"? The easy answer is that most mathematics starts in the real world. For instance, after observing on innumerable occasions that two sheep plus two more sheep make four sheep, ditto cows, wolves, warts and witches, it is a small step to introduce the idea that 2 + 2 = 4 in a universal, abstract sense. Since the abstraction came out of reality, it's no surprise if it applies to reality.

However, that is too simple-minded a view. Mathematics has an internal structure of logical deduction that allows it to grow in unexpected ways. New ideas can be generated internally too, whenever anyone tries to fill obvious holes in the logical landscape. For example, having worked out how to solve quadratic equations, which arose from problems about baking bread, or whatever, it is obvious that you ought to try to solve cubic and quintic equations too. Before you can say "Evariste Galois" you're doing Galois theory, which shows that you can't solve quintics, but is almost totally useless for anything practical. Then someone generalises Galois theory so that it applies to differential equations, and suddenly you find applications again, but to dynamics, not to bakery.

Herd of Elephants

Yes, there is a flow of problems and concepts from the real world into mathematics, and a back-flow of solutions from mathematics to reality. Wigner's point is that the back-flow may not answer the problem that you set out to solve. Instead it may answer something just as real, just as important, but physically unrelated.

Why should this be? Well, mathematics is the art of drawing necessary conclusions, independently of interpretations. Two plus two has to be four, whether you are discussing sheep, cows or witches. In other words, the same abstract structure can have several interpretations. So you can get the ideas from one interpretation, and transfer the result to others. Mathematics is so powerful because it is an abstraction.

This is all very well, but why do the abstractions of mathematics match reality? Indeed, do they really match, or is it all an illusion? Enter cultural relativism—the idea that has lately become so fashionable in academic arts departments, which sees maths and science as social constructs no less and no more valid than any other social construct. Does this lead to the idea that science can be anything scientists want it to be?

True, science is a social construct. Scientists who claim that it is not are making the same mistake as those who think that we all dip into the same collective subconscious. But there is something special about science: it is a construct that has at every step been tested against external reality. If the world's scientists all got together and said all elephants are weightless and rise into the air if they are not held down by ropes it would still be foolish to stand under a cliff when a herd of elephants was leaping off the edge. In science, there has to be a reality check. Because it is done by beings who see reality through imperfect and biased senses, the reality check cannot be perfect, but science still has to survive some very stringent scrutiny.

So what's the reality check in maths? Well, the deeper we delve into the "fundamental" nature of the Universe, the more mathematical it seems to get. The ghostly world of the quantum cannot be expressed without mathematics: if you try to describe it in everyday language, it makes no sense. Mind you, not all fields are so obviously mathematical in their structure. The biological world, in particular, seems not to obey the rigid rules that we find in physics. The "Harvard law of animal behaviour"—in carefully controlled laboratory conditions, animals do what they damned well please—is more appropriate than Newton's laws of motion. But the problem here could be a difference of scale. Quantum physics tends to be applied to simple arrangements of matter—a few atoms, say. In biology, the significant arrangements of matter are enormously more complex: there are trillions of atoms in the human genome, and this is just one DNA strand inside one cell of a much more complex organism. An atom-by-atom description of a human being would involve numbers with an awful lot of zeros. Human beings could well be behaving according to mathematical rules—but it is mathematics so complicated that human mathematicians cannot possibly write it down, let alone grasp what it means. Moreover, it is mathematics whose structure is almost totally impenetrable, for the boring reason that there is just too much information to take in.

This is the old philosophical problem of "emergence", but in a new guise. Emergent phenomena are things that seem to transcend their ingredients, like consciousness arising in a material brain. Philosophers have a habit of discussing emergence as if it breaks the chain of causality, but what really happens is the chain of causality becomes so intricate that the human mind cannot grasp it. Your behaviour is caused by mathematical rules applied to your constituent atoms, in the context of everything that is happening around you, but you can't do the calculations to check that because they're too messy and too lengthy.

You could argue that this makes the whole question academic: it doesn't matter whether this kind of mathematical basis exists for biology, because even if it does exist, it's of no practical use. However, there is an attractive alternative. Even very complex mathematical systems tend to generate recognizable patterns on higher levels of description. For example, the underlying quantum theory of a crystal involves just as many atoms as a human being, at least if it's a human sized crystal, and therefore runs into the same intractable problem of emergence. But crystals exhibit clear mathematical patterns of their own, such as a regular geometric form, and while nobody can deduce this in full logical rigour from the quantum mechanics of their atoms, there is a chain of reasoning that makes it plausible that the laws of quantum mechanics do indeed lead to the regularities of crystal structure. Roughly speaking, it goes like this: quantum mechanics causes the atoms to arrange themselves in a minimum-energy configuration; the overall symmetry of the laws of nature in space and time causes such configurations to be highly symmetrical; in this case, the consequence is that they form regular atomic lattices.

Lottery Illusion

From this point of view, mathematical patterns that arise in high-level descriptions of living organisms are evidence that biology, too, is mathematical at heart. For example, the number of petals in a flower tends to be one of the Fibonacci numbers— 3, 5, 8, 13, 21, 34, 55 and so on, where each is the sum of the previous two. This strange numerology can be traced to the dynamical behaviour of the cells at the tip of a growing shoot. The "primordia"—tiny lumps of cells from which the interesting features of plants develop—become arranged in patterns like interpenetrating spirals, and the mathematics of such patterns leads inevitably to Fibonacci numbers.

But do patterns like these really tell us that mathematics is inherent in nature? Our minds certainly have a tendency to seek out mathematical patterns, whether or not they are actually significant. This tendency has led to Newton's law of gravity and the equations of quantum mechanics, and also to astrology and an obsession with the measurements of the Great Pyramid. Ironically, what mathematics tells us about choosing lottery numbers is that any patterns we think we see are illusions.

It's worth asking how our minds developed this tendency for pattern seeking. Human minds evolved in the real world, and they learnt to detect patterns to help us survive events outside ourselves. If none of the patterns detected by these minds bore any genuine relation to the real world outside, they wouldn't have helped their owners survive, and would eventually have died out. So our figments must correspond, to some extent, to real patterns. In the same way, mathematics is our way of understanding certain features of nature. It is a construct of the human mind, but we are part of nature, made from the same kind of matter, existing in the same kinds of space and time as the rest of the Universe. So the figments in our heads are not arbitrary inventions. There are definitely some mathematical things in the Universe, the most obvious being the mind of a mathematician. Mathematical minds cannot evolve in an unmathematical universe. Only a geometer God can create beings able to come up with geometry.

But that is not to say that only one kind of mathematics is possible: the mathematics of the Universe. That seems too parochial a view. Would aliens necessarily come up with the same kind of mathematics as us? I don't mean in fine detail. For example the six-clawed cat creatures of Apellobetnees Gamma would no doubt use base-24 notation, but they would still agree that twenty-five is a perfect square, even if they write it as 11. However, I'm thinking more of the kind of mathematics that might be developed by the plasma vortex wizards of Cygnus V, for whom everything is in constant flux. I bet they'd understand plasma dynamics a lot better than we do, though I suspect we wouldn't have any idea how they did it. But I doubt that they would have anything like Pythagoras' theorem. There are few right angles in plasmas. In fact, I doubt they'd have the concept "triangle". By the time they had drawn the third vertex of a right triangle, the other two would be long gone, wafted away on the plasma winds.